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On non-conformal limit of the AGT relations

A. Marshakov[§], **A. Mironov**[¶]

Theory Department, Lebedev Physics Institute and *ITEP, Moscow, Russia*

A.Morozov[‖]

ITEP, Moscow, Russia

The Seiberg-Witten prepotentials for $\mathcal{N} = 2$ SUSY gauge theories with $N_f < 2N_c$ fundamental multiplets are obtained from conformal $N_f = 2N_c$ theory by decoupling $2N_c - N_f$ multiplets of heavy matter. This procedure can be lifted to the level of Nekrasov functions with arbitrary background parameters ϵ_1 and ϵ_2. The AGT relations imply that similar limit exists for conformal blocks (or, for generic $N_c > 2$, for the blocks in conformal theories with W_{N_c} chiral algebra). We consider the limit of the four-point function explicitly in the Virasoro case of $N_c = 2$, by bringing the dimensions of external states to infinity. The calculation is performed entirely in terms of representation theory for the Virasoro algebra and reproduces the answers conjectured in arXiv:0908.0307 with the help of the brane-compactification analysis and computer simulations. In this limit, the conformal block involving four external primaries, corresponding to the theory with vanishing beta-function, turns either into a 2-point or 3-point function, with certain coherent rather than primary external states.

1. The AGT relations [1]-[11] express generic $2d$ conformal blocks through the Nekrasov functions [12]-[20] $\mathcal{Z}(Y)$, associated with $\mathcal{N} = 2$ SUSY quiver $4d$ gauge theories with extra fundamental multiplets, generalizing the earlier predictions of [14, 15]. Most commonly these theories have vanishing beta-functions and possess conformal invariance in four dimensions. In the simplest case of the 4-point Virasoro conformal block, this is the conformal $SU(2)$ model with $N_f = 2N_c = 4$ flavors. The masses $\mu_1, \ldots, \mu_{N_f}$ of the four fundamentals are related to the dimensions of four external states operators:

$$\mu_1 = \alpha_1 - \alpha_2 + \frac{\epsilon}{2}, \quad \mu_2 = \alpha_1 + \alpha_2 - \frac{\epsilon}{2}, \quad \mu_3 = \alpha_3 - \alpha_4 + \frac{\epsilon}{2}, \quad \mu_4 = \alpha_3 + \alpha_4 - \frac{\epsilon}{2},$$
$$\Delta_k = \frac{\alpha_k(\epsilon - \alpha_k)}{\epsilon_1\epsilon_2}, \qquad c = 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}, \qquad \epsilon = \epsilon_1 + \epsilon_2 \tag{1}$$

and the gauge theory condensate (modulus) $a = a_1 = -a_2$ is related to that of the intermediate state:

$$a = \alpha - \frac{\epsilon}{2} \tag{2}$$

For large masses $\mu_k \to \infty$ the fundamental fields in $4d$ theory decouple, and one gets an asymptotically free pure gauge $\mathcal{N} = 2$ SUSY theory, with prepotential expressed through (the $\epsilon_1 = -\epsilon_2 \to 0$ limit of) the *pure gauge* Nekrasov functions $Z(Y)$:

$$Z(Y) \sim \lim_{\mu_k \to \infty} \mathcal{Z}(Y) \tag{3}$$

The AGT relation implies that the associated limit of conformal block corresponds to this $Z(Y)$. A natural question is how does this limit look like from the point of view of $2d$ conformal theory itself.

[§]E-mail: mars@itep.ru; mars@lpi.ru
[¶]E-mail: mironov@itep.ru; mironov@lpi.ru
[‖]E-mail: morozov@itep.ru

This question was addressed in [4] and an elegant answer has been proposed: the relevant conformal blocks are matrix elements for certain "coherent" states in Verma module of Virasoro algebra. However, in [4] the answer was not derived in a *direct* way, by taking a particular limit of the 4-point conformal block with generic μ's. Instead, the conclusion was based on analysis of the underlying 5-brane configurations [21], which was also the original source of the AGT relations in [1]. In this letter, we fill the gap and derive the result of [4] straightforwardly, making use of explicit knowledge of the Virasoro conformal blocks from [3]. A similar analysis is possible for conformal blocks with more external states and for *some* W-algebra blocks $N_c > 2$, in the last case the results of [2, 5, 7] should be used. These generalizations are, however, beyond the scope of this paper.

2. We use notations from [3] and refer for details and explanations to that paper. The 4-point conformal block is given by the sum over Young diagrams

$$\mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x) = \sum_{|Y|=|Y'|} x^{|Y|}\gamma_{\Delta\Delta_1\Delta_2}(Y) Q_\Delta^{-1}(Y,Y')\gamma_{\Delta\Delta_3\Delta_4}(Y') \tag{4}$$

with the inverse Shapovalov form $Q_\Delta(Y,Y') = \langle\Delta|L_{Y'}L_{-Y}|\Delta\rangle$, where $L_{-Y} = L_{-k_\ell}\dots L_{-k_2}L_{-k_1}$ for the Young diagram $Y = \{k_1 \geq k_2 \geq \dots \geq k_\ell > 0\}$ are made from the Virasoro operators L_k, $k \in \mathbb{Z}$, satisfying

$$[L_m, L_n] = \frac{c}{12}n(n^2-1)\delta_{m+n,0} + (m-n)L_{m+n} \tag{5}$$

and the three-point functions [22, 3] are

$$\gamma_{\Delta\Delta_1\Delta_2}(Y) = \prod_{i=1}^{\ell(Y)}\left(\Delta + k_i\Delta_1 - \Delta_2 + \sum_{j<i} k_j\right) \tag{6}$$

The Shapovalov matrix $Q_\Delta(Y,Y') = Q_\Delta([k_1k_2\dots],[k_1'k_2'\dots])$ is infinite-dimensional, but has an obvious block form, since the matrix elements are non-vanishing only when $|Y| = |Y'|$. Therefore, for generic Δ and c, it is straightforwardly invertible. The AGT relations [1, 3] state that, under the identification (1) and (2),

$$\mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x) = \mathcal{Z}(x) = \sum_Y x^{|Y|}\mathcal{Z}(Y) \tag{7}$$

and we are going to turn now to the asymptotically free limit of this relation.

3. We would like to consider first the limit of conformal block (4), when all $\mu_1, \dots, \mu_4 \to \infty$ independently, and, at the same time, $x \to 0$ so that

$$x\prod_{I=1}^{4}\mu_I = \Lambda^4 \tag{8}$$

which is a scale (like Λ_{QCD}) parameter, in the pure $\mathcal{N} = 2$ SUSY gauge theory with $N_f = 0$. For this we do not even need an explicit form of the Shapovalov matrix, since it does not depend on external dimensions $\Delta_{1,2,3,4}$.

However, explicit formula (6) is crucially important. The number of factors in the r.h.s. of (6) is equal to the number of rows $\ell(Y)$ (the number of non-vanishing k's) in the Young diagram Y, and it is maximal for fixed $|Y|$ when the diagram consists of a single column, i.e. when all $k_i = 1$, $1 \leq i \leq \ell(Y)$ or $\ell(Y) = |Y|$.

Since in our limit $\Delta_i \gg \Delta, 1$, the γ-factor reduces to

$$\gamma(Y) \sim \prod_{i=1}^{\ell(Y)}(k_i\Delta_1 - \Delta_2) \tag{9}$$

and of all diagrams of a given size $|Y|$, the sum in (4) is saturated by the terms, where $\gamma(Y)$'s (9) contain maximal possible number of factors, i.e. when $\ell(Y) = |Y|$, or Y is a single-column diagram $[1^{|Y|}] = [\underbrace{1,\ldots,1}_{|Y|\ \text{times}}]$:

$$
\begin{aligned}
&x^{|Y|/2}\gamma_{\Delta\Delta_1\Delta_2}(Y) \to \Big(\sqrt{x}\,(\Delta_1-\Delta_2)\Big)^{|Y|}\delta(Y,[1^{|Y|}]) = \\
&= \left(\frac{\sqrt{x}\mu_1\mu_2}{-\epsilon_1\epsilon_2}\right)^{|Y|}\delta(Y,[1^{|Y|}]) \to \left(\frac{\Lambda^2}{-\epsilon_1\epsilon_2}\right)^{|Y|}\delta(Y,[1^{|Y|}])
\end{aligned}
\tag{10}
$$

In what follows we often omit the powers of $-\epsilon_1\epsilon_2$, which can be easily restored from dimensional consideration. Since the Shapovalov form does not depend on $\Delta_1,\ldots,\Delta_4$, this means that the limit of conformal block

$$
\boxed{B_\Delta(\Lambda) = \lim_{\Delta_i\to\infty}\mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x) = \sum_{|Y|=|Y'|}\Lambda^{4|Y|}Q_\Delta^{-1}(Y,Y')\delta\Big(Y,[1^{|Y|}]\Big)\delta\Big(Y',[1^{|Y|'}]\Big) = \sum_n \Lambda^{4n}Q_\Delta^{-1}([1^n],[1^n])}
\tag{11}
$$

The r.h.s. of this expression can be treated as a norm (scalar square) of a peculiar vector in the Virasoro Verma module $\mathcal{H}_\Delta$ with the highest weight Δ. Following [4], we denote it $|\Lambda^2,\Delta\rangle = \sum_Y C_Y L_{-Y}|\Delta\rangle \in \mathcal{H}_\Delta$. Then

$$
\||\Delta,\Lambda^2\rangle\|^2 = \langle\Delta,\Lambda^2|\Delta,\Lambda^2\rangle = \sum_{Y,Y'} C_Y Q(Y,Y')C_{Y'}
\tag{12}
$$

and, in order to reproduce the r.h.s. of (11), one should take $C_Y = \Lambda^{|Y|}Q_\Delta^{-1}([1^{|Y|}],Y)$, so that

$$
\boxed{|\Delta,\Lambda^2\rangle \ = \sum_Y \Lambda^{2|Y|}Q_\Delta^{-1}\Big([1^{|Y|}],Y\Big)L_{-Y}|\Delta\rangle}
\tag{13}
$$

This vector can be characterized as being orthogonal to all non single-column states $|\Delta,Y\rangle = L_{-Y}|\Delta\rangle \in \mathcal{H}_\Delta$ with $Y \neq [1^{|Y|}]$, since

$$
\begin{aligned}
&\langle\Delta|L_Y|\Delta,\Lambda^2\rangle = \sum_{Y'}\Lambda^{2|Y'|}Q_\Delta^{-1}\Big([1^{|Y'|}],Y'\Big)\langle\Delta|L_Y L_{-Y'}|\Delta\rangle = \\
&= \sum_{Y'}\Lambda^{2|Y'|}Q_\Delta^{-1}\Big([1^{|Y'|}],Y'\Big)Q_\Delta(Y',Y) = \Lambda^{2|Y|}\delta\Big(Y,[1^{|Y|}]\Big)
\end{aligned}
\tag{14}
$$

This means, in particular, that it is a kind of a "coherent" state, satisfying

$$
\begin{aligned}
L_1|\Delta,\Lambda^2\rangle &= \Lambda^2|\Delta,\Lambda^2\rangle, \\
L_k|\Delta,\Lambda^2\rangle &= 0, \quad \forall \quad k\geq 2
\end{aligned}
\tag{15}
$$

The implication (14)⇒(15) deserves more detailed explanation. Consider the vector $L_k|\Delta,\Lambda^2\rangle \in \mathcal{H}_\Delta$ for $k>0$. The coefficients of its expansion over the basis $|\Delta,Y\rangle = L_{-Y}|\Delta\rangle$ in $\mathcal{H}_\Delta$ are characterized totally by the scalar products

$$
\begin{aligned}
&\langle\Delta,Y|L_k|\Delta,\Lambda^2\rangle = \langle\Delta|L_Y L_k|\Delta,\Lambda^2\rangle = \sum_{Y'} b^{(k)}_{YY'}\langle\Delta|L_{Y'}|\Delta,\Lambda^2\rangle \underset{(14)}{=} \\
&= \sum_{Y'} b^{(k)}_{YY'}\Lambda^{2|Y'|}\delta\Big(Y',[1^{|Y'|}]\Big) = \sum_{\ell'} b^{(k)}_{Y[1^{\ell'}]}\Lambda^{2\ell'}
\end{aligned}
\tag{16}
$$

where $\ell' = \ell(Y') = |Y'|$, i.e. only the Young diagrams $Y' = [1^{|Y'|}] = [1^{\ell(Y')}]$ can contribute. It is important, however, that due to the Virasoro commutation relations, (5) the sum in (16) is restricted by $|Y'| \leq |Y| + k$ and $\ell(Y') \leq \ell(Y)+1$, meaning that both the number of boxes in Y' and the number

of elementary Virasoro generators in $L_{Y'}$ is less or equal to those in $L_Y L_k$; moreover, the structure of Virasoro algebra (5) requires necessarily $k_i(Y') \geq k_j(Y)$, $i,j = 1,\dots,\ell(Y'),\ell(Y)$. Hence, one gets for (16)

$$\langle\Delta, Y|L_k|\Delta,\Lambda^2\rangle = \sum_{\ell'} b^{(k)}_{Y[1^{\ell'}]}\Lambda^{2\ell'} = \delta\Big(Y,[1^\ell]\Big)\sum_{\ell'\leq\ell+1} b^{(k)}_{[1^\ell][1^{\ell'}]}\Lambda^{2\ell'} = \delta\Big(Y,[1^\ell]\Big)\delta_{k,1}\Lambda^{2\ell+2} \tag{17}$$

and this immediately leads to (15), since for $k>1$ the vector $L_k|\Delta,\Lambda^2\rangle$ is orthogonal to all vectors in $\mathcal{H}_\Delta$, while for $k=1$ it coincides with the vector $|\Delta,\Lambda^2\rangle$ up to a numerical factor Λ^2.

Differently, expanding $|\Delta,\Lambda^2\rangle = \sum_{n\geq 0}\Lambda^{2n}|\Delta,n\rangle$, one gets for

$$|\Delta,n\rangle = \sum_{|Y|=n} Q_\Delta^{-1}\Big([1^n],Y\Big)L_{-Y}|\Delta\rangle \tag{18}$$

that

$$\begin{aligned} L_1|\Delta,n\rangle &= |\Delta,n-1\rangle, \quad n\geq 0 \\ L_k|\Delta,n\rangle &= 0, \quad \forall \quad k\geq 2,\ n\geq 0 \end{aligned} \tag{19}$$

which is exactly the claim of [4]. Here we have derived and *proved* it directly, taking the limit of the 4-point conformal block with arbitrary dimensions. Note that the whole reasoning is valid for any ϵ_1,ϵ_2 and ϵ, i.e. for conformal theory with arbitrary central charge c (the central charge dependence arises in $|\Delta,\Lambda^2\rangle$ through the inverse matrix of the Shapovalov form).

5. In a similar way, one can consider partial decoupling of the fundamental matter, corresponding to the models with $N_f = 1,2,3$, when remaining masses (and related combinations of conformal dimensions) are preserved as free parameters. Let us start with the case of $N_f=1$. In such a limit, $\mu_{2,3,4}\to\infty$ with finite $x\prod_{I=2,3,4}\mu_I = \Lambda_1^3$, but μ_1 remains finite itself. According to (1), this means that α_1 and α_2 go to infinity, but not independently: their difference remains finite. In terms of conformal dimensions it means that

$$\Delta_1-\Delta_2 = \frac{(\alpha_1-\alpha_2)(\alpha_1+\alpha_2-\epsilon)}{-\epsilon_1\epsilon_2} \sim \frac{(2\mu_1-\epsilon)\sqrt{\Delta_1}}{\sqrt{-\epsilon_1\epsilon_2}} \tag{20}$$

i.e. all dimensions are infinite, but $\frac{\Delta_1-\Delta_2}{\sqrt{\Delta_1}}$ remains finite. Hence, in this limit the single-column diagrams still dominate to contribute into $\gamma_{\Delta\Delta_3\Delta_4}(Y)$, like it has been considered above in the case of flow into pure gauge theory, but the factor $\gamma_{\Delta\Delta_1\Delta_2}(Y)$ is now dominated by a different sort of Young diagrams. The reason is that for $k_i=1$ the factor $k_i\Delta_1-\Delta_2$ turns into $\Delta_1-\Delta_2$ and grows not as fast as Δ_1 and Δ_2 themselves. Instead, the dominant contribution comes now from the Young diagrams of the form $Y=[2^p,1^q]$ with $|Y|=2p+q$, $\ell(Y)=p+q$, since for all of them

$$\gamma_{\Delta\Delta_1\Delta_2}(Y) \sim (2\Delta_1-\Delta_2)^p(\Delta_1-\Delta_2)^q \sim \left(\frac{2\mu_1-\epsilon}{\sqrt{-\epsilon_1\epsilon_2}}\right)^q\Delta_1^{p+q/2} \sim \frac{(2\mu_1-\epsilon)^q(\mu_2/2)^{|Y|}}{(-\epsilon_1\epsilon_2)^{p+q}} \tag{21}$$

Instead of (11), the limit of conformal block is now given by (we again omit the powers of $-\epsilon_1\epsilon_2$)

$$\begin{aligned} B_\Delta^{N_f=1}(\Lambda_1,m) &= \lim_{\substack{\Delta_I\to\infty \\ \Delta_1-\Delta_2\sim 2m\sqrt{\Delta_1}}} \mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x) = \\ &= \sum_{|Y|=|Y'|}\sum_p (2m)^{|Y|-2p}\left(\frac{x\mu_2\mu_3\mu_4}{2}\right)^{|Y|} Q_\Delta^{-1}(Y,Y')\delta\Big(Y,[2^p,1^{|Y|-2p}]\Big)\delta\Big(Y',[1^{|Y|'}]\Big) = \\ &= \sum_{n,p}(2m)^{n-2p}\left(\frac{\Lambda_1^3}{2}\right)^n Q_\Delta^{-1}\Big([2^p,1^{n-2p}],[1^n]\Big) = \langle\Delta,\Lambda_1/2,2m|\Delta,\Lambda_1^2\rangle, \end{aligned} \tag{22}$$

where $m=\mu_1-\frac{\epsilon}{2}$, $\Lambda_1^3=x\mu_2\mu_3\mu_4$ to be fixed when taking the limit of $x\to 0$ and $\mu_I\to\infty$, $I=2,3,4$, and

$$\boxed{|\Delta,\Lambda,m\rangle=\sum_Y\sum_p m^{|Y|-2p}\Lambda^{|Y|}Q_\Delta^{-1}\Big([2^p,1^{|Y|-2p}],\ Y\Big)L_{-Y}|\Delta\rangle} \tag{23}$$

while the vector $|\Delta,\Lambda^2\rangle$ has been already defined in (13). Considering the matrix elements

$$\langle\Delta|L_Y|\Delta,\Lambda,m\rangle\ =\sum_p m^{|Y|-2p}\Lambda^{|Y|}\delta\Big(Y,[2^p,1^{|Y|-2p}]\Big) \tag{24}$$

and

$$\langle\Delta|L_YL_k|\Delta,\Lambda,m\rangle\ =\sum_{Y'}b^{(k)}_{YY'}\langle\Delta|L_{Y'}|\Delta,\Lambda,m\rangle\ \underset{(24)}{=}\ \sum_{Y'}b^{(k)}_{YY'}\sum_p m^{|Y'|-2p}\Lambda^{|Y'|}\delta\Big(Y',[2^p,1^{|Y'|-2p}]\Big)=$$

$$=\delta_{k,1}b^{(1)}_{[2^p,1^{|Y|-2p}][2^p,1^{|Y|+1-2p}]}m^{|Y|+1-2p}\Lambda^{|Y|+1}+\delta_{k,2}b^{(2)}_{[2^p,1^{|Y|-2p}][2^{p+1},1^{|Y|+2-2(p+1)}]}m^{|Y|+2-2(p+1)}\Lambda^{|Y|+2} \tag{25}$$

one proves exactly in the same way as before that

$$\begin{aligned} L_1|\Delta,\Lambda,m\rangle&=m\Lambda|\Delta,\Lambda,m\rangle,\\ L_2|\Delta,\Lambda,m\rangle&=\Lambda^2|\Delta,\Lambda,m\rangle,\\ L_k|\Delta,\Lambda,m\rangle&=0\qquad\text{for}\quad k\geq 3\end{aligned} \tag{26}$$

again in agreement with the claim of [4].

Note also that, in the limit when $m\to\infty$ together with $\Lambda\to 0$ so that $m\Lambda=\Lambda^2_{N_f=0}$, only the term with $p=0$ survives in the sum (23) and this state turns into (13): $|\Delta,m,\Lambda\rangle\to|\Delta,\Lambda^2_{N_f=0}\rangle$, while constraints (26) turn into (15). It deserves mentioning that, due to separation of powers of Λ_1 in (22) between two vectors in the scalar product (which is, of course, ambiguous), this limit is a little bit different from the conventional "physical" limit in Seiberg-Witten theory, $\mu_1\Lambda_1^3\to\Lambda^4_{N_f=0}$.

The calculation is very similar in the case of $N_f=2$, if keeping finite the masses μ_1 and μ_3. Then, both the factors $\gamma_{\Delta\Delta_1\Delta_2}$ and $\gamma_{\Delta\Delta_3\Delta_4}$ behave according to (21), when taking $\mu_2\to\infty$ and $\mu_4\to\infty$, and one gets that the conformal block (4)

$$B_\Delta^{N_f=2}(\Lambda_2,m_1,m_3)=\lim_{\substack{\Delta_I\to\infty\\ \Delta_1-\Delta_2\sim 2\mu_1\sqrt{\Delta_1}\\ \Delta_3-\Delta_4\sim 2\mu_3\sqrt{\Delta_3}}}\mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x)=$$

$$=\sum_{|Y|=|Y'|}\sum_{p,p'}(2m_1)^{|Y|-2p}(2m_3)^{|Y|-2p'}\left(\frac{\Lambda_2}{2}\right)^{2|Y|}Q_\Delta^{-1}(Y,Y')\delta\Big(Y,[2^p,1^{|Y|-2p}]\Big)\delta\Big(Y',[2^{p'},1^{|Y'|-2p'}]\Big)=$$

$$=\sum_{n,p,p'}(2\mu_1)^{n-2p}(2\mu_3)^{n-2p'}\left(\frac{\Lambda_2}{2}\right)^{2n}Q_\Delta^{-1}\Big([2^p,1^{n-2p}],[2^{p'},1^{|Y'|-2p'}]\Big)=$$

$$=\langle\Delta,\Lambda_2/2,2m_1|\Delta,\Lambda_2/2,2m_3\rangle \tag{27}$$

in this limit is a scalar product of two states (23), where $m_{1,3}=\mu_{1,3}-\frac{\epsilon}{2}$, $\Lambda_2^2=x\mu_2\mu_4$, are again to be fixed finite in the limit of $x\to 0$ and $\mu_{2,4}\to\infty$.

6. In the case of "asymmetric limit", i.e. if instead of taking $\mu_{2,4}\to\infty$, one decouples, say, $\mu_{3,4}\to\infty$, no simplification occurs in the factor $\gamma_{\Delta\Delta_1\Delta_2}(Y)$ in (4), while the second factor degenerates according to (10), i.e. $x^{|Y'|}\gamma_{\Delta\Delta_3\Delta_4}(Y')\to\Lambda^{2|Y'|}\delta\Big(Y',[1^{|Y'|}]\Big)$. This means that the conformal block

simplifies, though not as drastically as in the symmetric limit:

$$\tilde{B}^{N_f=2}_{\Delta}(\Lambda_2,\mu_1,\mu_2) = \lim_{\Delta_{3,4}\to\infty}\mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x) = \sum_Y \Lambda^{2|Y|}\gamma_{\Delta\Delta_1\Delta_2}(Y)Q_\Delta^{-1}\Big(Y,[1^{|Y|}]\Big) = \\ = \langle\Delta,\Lambda^2|V_{\Delta_1}(1)V_{\Delta_2}(0)\rangle \tag{28}$$

since [3][1]

$$\gamma_{\Delta\Delta_1\Delta_2}(Y) = \langle L_{-Y}V_\Delta|V_{\Delta_1}(1)V_{\Delta_2}(0)\rangle \tag{29}$$

Thus, the 4-point conformal block in this limit reduces to a triple vertex, as was conjectured in [4]. It depends on μ_1 and μ_2 through Δ_1 and Δ_2.

Similarly, if only one mass, say, $\mu_4\to\infty$, one obtains

$$B^{N_f=3}_{\Delta}(\Lambda_3,\mu_1,\mu_2,\mu_3) = \lim_{\substack{\Delta_{3,4}\to\infty \\ \Delta_3-\Delta_4\sim 2\mu_3\sqrt{\Delta_3}}}\mathcal{B}_{\Delta_1\Delta_2;\Delta_3\Delta_4;\Delta}(x) = \\ = \sum_Y\sum_p (2\mu_3-\epsilon)^{|Y|-2p}\left(\frac{\Lambda}{2}\right)^{|Y|}\gamma_{\Delta\Delta_1\Delta_2}(Y)Q_\Delta^{-1}\Big(Y,[2^p,1^{|Y|-2p}]\Big) = \\ = \langle\Delta,\Lambda/2,2\mu_3-\epsilon|V_{\Delta_1}(1)V_{\Delta_2}(0)\rangle \tag{30}$$

which is again a reduction from the 4-point function to a 3-point one.

7. To conclude, in this paper we have studied the non-conformal limits (in the sense of $4d$ supersymmetric gauge theory) of conformal blocks related to Nekrasov partition functions by the AGT correspondence. We have derived directly from $2d$ CFT analysis the results, conjectured in [4] from brane considerations and confirmed by computer simulations, for the asymptotically free limit of conformal blocks. The proof holds at the level of Nekrasov functions for arbitrary values of ϵ_1,ϵ_2 and $\epsilon=\epsilon_1+\epsilon_2$, the result for the Seiberg-Witten prepotentials [23] follows [15, 16] after taking the limit of $\epsilon_1,\epsilon_2\to 0$. The proof is self-consistent within $2d$ CFT, and, in application to Nekrasov functions, it assumes that the original AGT relation is correct. After numerous checks in [1]-[11] this looks indisputably true, though so far has been proven exactly [6, 7] only in the hypergeometric case for the W-algebra blocks with one special, one fully-degenerate external state and a free field theory like selection rule imposed on the intermediate state.

There is a number of other interesting limits, which are natural and well understood from the point of view of $2d$ CFT (e.g. large intermediate dimension Δ or the central charge c). It can be interesting to find their interpretation in terms of the Nekrasov functions and/or instanton expansions in $4d$ SUSY models.

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